Suppression of quantum-radiation-pressure noise in an optical spring

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Recent advances in micro- and nanofabrication techniques have led to corresponding improvement in the performance of optomechanical systems, which provide a promising avenue towards quantum-limited metrology and the study of quantum behavior in macroscopic mechanical objects. One major impediment to reaching the quantum regime is thermal excitation, which can be overcome for a sufficiently high mechanical quality factor Q. Here, we propose a method for increasing the effective Q of a mechanical resonator by stiffening it via the optical spring effect exhibited by linear optomechanical systems and show how the associated quantum-radiation-pressure noise can be evaded by sensing and feedback control. In a parameter regime that is attainable with current technology, this method allows for realistic quantum cavity optomechanics in a frequency band well below that which has been realized thus far.

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I. INTRODUCTION

Catalyzed by vast improvements in micro- and nanofabrication processes, the field of cavity optomechanics has seen a recent boom in interest [1–3]. In addition to providing a means for quantum-limited force measurements [4], e.g., in gravitational-wave detection [5] and scanning probe microscopy, optomechanical devices can also be used to probe the quantum behavior of mechanical systems. Recently, several experiments have demonstrated the cooling of a resonator down to its quantum ground state via cryogenics or optomechanical interaction [6–8]. In addition, more than one group [9,10] has demonstrated the so-called optomechanically induced transparency effect, an analog of the electromagnetically induced transparency [11,12] effect observed in atomic systems. This effect can be used to make narrow-band quantum filters, e.g., to effect the frequency-dependent phase rotation of squeezed light injected to enhance the sensitivity of quantum noise-limited interferometric gravitational wave detectors [13,14]. This also opens up the possibility for the processing and storing of nonclassical states of light through coherent transfer of quantum states between light and a mechanical oscillator, a technique that would find much use in the emergent field of quantum information processing.

The ubiquitous bath of thermal energy presents a major obstacle to these efforts, randomly exciting a system and masking its underlying quantum nature. A characteristic figure of merit for quantifying this thermal decoherence effect is given by the ratio of the thermal occupation number $\bar{n}_{\rm th}$ and the mechanical quality factor Q:

$$\frac{\bar{n}_{\rm th}}{Q} = \frac{k_B T}{\hbar \omega_m Q} \propto (Qf)^{-1}, \tag{1}$$

where k_B is the Boltzmann's constant and $f = \omega_m/2\pi$ is the mechanical frequency. When this ratio becomes <1, the oscillator quantum state will survive longer than one oscillation period before the thermal effect destroys it.

As is apparent from Eq. (1), the quantum-state lifetime is ultimately limited by the product of the quality factor Q and mechanical frequency f. A significant body of research has focused on increasing this "Qf product" for a wide range of mechanical systems. If Q were truly a frequency-independent quantity—as in the "structural damping" model as described by Saulson [15]—then moving to higher eigenfrequencies would lead to an immediate improvement. In the opposite direction, there are many experiments that would benefit from the use of low-frequency (sub-kHz) resonators. A number of bulk structures have been found to exhibit extremely high Q values in this frequency range [16,17]; unfortunately, such systems tend to have relatively large (gram- to kilogram-scale) effective masses, making them unsuitable for typical optomechanics experiments. The realization of submicrogram effective masses requires the use of nanofabricated resonators. In practice, excess damping from surface effects [18], phonon tunneling loss [19], or intrinsic mechanisms such as thermoelastic [20] and Akhiezer [21] damping limits the achievable Q and thus the Qf product in these devices. In addition, we add the further requirement that the desired system exhibit excellent optical quality (i.e., high reflectivity owing to low scatter loss and absorption), which limits the resonator options considerably, especially in light of the fact that typical dielectric materials used to create multilayer optical coatings (e.g., SiO_2/Ta_2O_5) exhibit low mechanical quality factors [22]. Here, we propose a method for using the optical spring effect in linear optomechanical devices [23–28] to increase the effective Q of a given mechanical resonator, while simultaneously suppressing the quantum-radiation-pressure noise that would normally be imparted by the optical fields. This technique should facilitate the creation of an oscillator with a Qf product considerably higher than those available today, enabling useful applications in quantum metrology and also creation of long-lived quantum states at lower frequencies than were previously practical.

The concept in this paper makes use of the fact that when a strong optical spring is linearly coupled to a mechanical resonator, the resonator's Hamiltonian becomes augmented or even dominated by contributions from the radiation pressure forces of the optical fields. In this way, the bare resonator's

thermal noise is "diluted" by the ratio of the intrinsic elastic energy to that stored in the optical field [27]. Typically, the modification of a resonator's dynamics via linear coupling is accompanied by excess noise from quantum back-action: the quantum fluctuations of the radiation pressure, in our case. This has been identified as a serious issue in the strong-dilution regime by Chang *et al.* [29] and Ni *et al.* [30], who instead propose to achieve optical dilution by using a nonlinear quadratic optical potential to trap a partially reflective membrane [31], which would be immune to linear quantum back-action. The device we propose evades such parasitic quantum back-action by detecting it in the outgoing field and actively feeding back to the system, resulting in a nearly noise-free optical spring. Since this method allows for straightforward coupling of the diluted mechanical resonator to an external optical system from the reverse side of the resonator, it can be used as a "black-box" effective mechanical resonator of exceptionally high quality.

While superficially similar to another intensity feedback scheme described by Buchler *et al.* [32], the technique described here differs in at least two important ways. For one, the critical coupling of the optical cavity employed in that previous work ensures that only half of the fluctuations responsible for the radiation pressure noise are measured (leading to a maximal ideal suppression factor of 2), while in our case all fluctuations are measured, leading to an unbounded suppression factor in the ideal case. More fundamentally, their technique is not applicable to a detuned optical cavity and is, therefore, unsuitable for use with an optical spring.

II. OPTICAL SPRING

The canonical optomechanical system is shown in the dashed box in Fig. 1. In such a system, the "optical spring" effect arises from dynamical back-action of the optical cavity field on the mechanical oscillator forming one cavity boundary. The mechanical oscillator displacement $\hat{x}$ is coupled to the cavity field $\hat{a}$ via radiation pressure, as described by the following interaction Hamiltonian [33]:

$$\hat{\mathscr{H}}_{\rm int} = \hbar G_0 \hat{x}(\bar{a}^* \hat{a} + \bar{a}\hat{a}^\dagger) \equiv -\hat{x}\hat{F}_{\rm rad}. \tag{2}$$

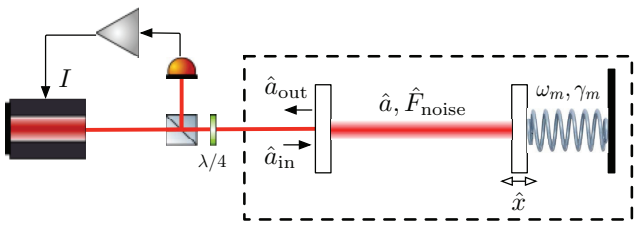

FIG. 1. (Color online) Simplified experimental layout, with the canonical optomechanical system shown within the dashed box. Input vacuum fluctuations drive the cavity mode, which in turn exerts radiation pressure forces on the mechanical resonator forming the cavity boundary. The output mode of the cavity is sensed with a photodetector, and—in the relevant parameter regime—the measured power contains the radiation pressure fluctuations that drive the resonator with minimal sensitivity to the resonator position. This signal can therefore be used as an error signal for feeding back to the laser amplitude to suppress the radiation pressure noise on the resonator.

The coupling constant is $G_0 = \omega_c/L$, $\bar{a}$ is the classical mean amplitude of $\hat{a}$ due to coherent driving of an external laser, ω_c is the cavity resonant frequency, and L is the cavity length. When the frequency of the external laser ω_0 that drives the cavity field is detuned from ω_c, $\hat{F}_{\rm rad}$ depends on the oscillator displacement, creating a mechanical response that mimics a spring. More specifically, $\hat{F}_{\rm rad}$ in the frequency domain can be written as (see the Appendix for a more detailed derivation)

$$\hat{F}_{\rm rad}(\omega) = -K_{\rm os}(\omega)\hat{x}(\omega) + \hat{F}_{\rm noise}(\omega), \tag{3}$$

where the optical spring coefficient $K_{\rm os}$ is approximately given by

$$\begin{aligned} K_{\rm os} &\approx -\frac{2\hbar G_0^2|\bar{a}|^2\Delta}{\Delta^2+\gamma^2} - \frac{4i\hbar G_0^2|\bar{a}|^2\gamma\Delta\omega}{(\Delta^2+\gamma^2)^2} \\ &\equiv m\omega_{\rm os}^2 - i\,m\Gamma_{\rm os}\omega, \end{aligned} \tag{4}$$

with the cavity detuning $\Delta \equiv \omega_c - \omega_0$ and γ being the cavity bandwidth. Here, the approximation is taken for the case of large detuning and cavity bandwidth, which we show to be the relevant parameter regime for realization of this idea. In addition, we have introduced the optical spring frequency $\omega_{\rm os}$ and the optical damping $\Gamma_{\rm os}$. As we can see, when the detuning is negative, i.e., $\Delta < 0$, the optical rigidity is real and positive, and the optical damping is negative $\Gamma_{\rm os}$ (heating), and vice versa. By introducing an additional driving field with a different detuning frequency, one can create the so-called stable double optical spring [26], which exhibits both positive rigidity and positive damping (we elaborate on this issue later). The optical spring modifies the mechanical susceptibility $\chi_0(\omega)$, defined through $\chi_0(\omega) \equiv \hat{x}(\omega)/\hat{F}(\omega)$, from its original value $\chi_0^{-1}(\omega) = -m(\omega^2 + i\gamma_m\omega - \omega_m^2)$ to an effective one:

$$\chi_{\rm eff}^{-1}(\omega) = -m\left[\omega^2 + i(\gamma_m + \Gamma_{\rm os})\omega - \left(\omega_m^2 + \omega_{\rm os}^2\right)\right]. \tag{5}$$

For a strong optical spring $\omega_{\rm os} \gg \omega_m$, we can significantly stiffen the mechanical oscillator with the restoring energy from the optical field.

One immediate issue with this approach comes from the quantum-radiation-pressure noise $\hat{F}_{\rm noise}(\omega)$ in Eq. (3), which arises from quantum fluctuation of the optical field,

$$\begin{aligned} \hat{F}_{\rm noise}(\omega) \equiv\ & \frac{2\hbar G_0|\bar{a}|\sqrt{\gamma}}{\sqrt{\gamma^2+\Delta^2}} \\ & \times\left[\frac{(\gamma^2+\Delta^2-i\,\gamma\,\omega)\hat{v}_1 + i\,\Delta\,\omega\,\hat{v}_2}{(\omega-\Delta+i\gamma)(\omega+\Delta+i\gamma)}\right], \end{aligned} \tag{6}$$

where $\hat{v}_1 \equiv (\hat{a}_{\rm in} + \hat{a}_{\rm in}^\dagger)/\sqrt{2}$ and $\hat{v}_2 \equiv (\hat{a}_{\rm in} - \hat{a}_{\rm in}^\dagger)/\sqrt{2}i$ are the amplitude and phase quadratures of the input optical field. This additional noise term will increase the effective temperature of the thermal bath and drive the mechanical oscillator away from the quantum regime, as pointed out by Chang *et al.* [29]. In the large bandwidth and detuning regime, this reduces to

$$\hat{F}_{\rm noise}(\omega) \approx -\frac{2\hbar G_0|\bar{a}|\sqrt{\gamma}}{\sqrt{\gamma^2+\Delta^2}}\hat{v}_1(\omega), \tag{7}$$

indicating that the radiation pressure noise is dominated by fluctuations in the amplitude quadrature of the input field. The

strength of this noise can be quantified by its spectral density:

$$S_F(\omega) \approx \frac{4\hbar^2 G_0^2 |\bar{a}|^2 \gamma}{\gamma^2 + \Delta^2}. \tag{8}$$

From the above expression and Eq. (4), we learn that the optical rigidity (real part of K_{os}) scales with the optomechanical coupling strength in the same way as the radiation pressure noise:

$$K_{os}, S_F \propto G_0^2 |\bar{a}|^2. \tag{9}$$

Essentially, this means that an increase in the optical spring frequency is accompanied by an increase in the radiation pressure noise when we scale up the optical power.

III. EVADING QUANTUM-RADIATION-PRESSURE NOISE

To solve the aforementioned issue, we make use of the fact that the output field emerging from the cavity contains information about the quantum-radiation-pressure noise that has been imposed onto the mechanical oscillator. In particular, as we show, for the large bandwidth and detuning limit, the power fluctuations in the output field originate from the same quadrature that is responsible for the radiation pressure noise. The photodetector measures these fluctuations, and by feeding this signal back to the mechanical oscillator with the correct filter, we can therefore evade the quantum-radiation-pressure noise and achieve a nearly noiseless optical spring. Note that this does not violate the fundamental principle of quantum measurement—any linear continuous measurement of a dynamical variable that does not commute at different times (nonconservative) is associated with quantum back-action on that variable [4]; here, we only sense the quantum-radiation-pressure noise and have almost no sensitivity to the mechanical displacement, and that is why we can evade such back-action noise.

To elaborate on this idea, we use the standard input-output relation for this system, $\hat{a}_{out}(\omega) = -\hat{a}_{in}(\omega) + \sqrt{2\gamma}\,\hat{a}(\omega)$, and it, for high bandwidth and detuning, gives (refer to the Appendix for more detail)

$$\hat{a}_{out}(\omega) \approx -\frac{\Delta + i\gamma}{\Delta - i\gamma}\hat{a}_{in}(\omega) - \frac{\sqrt{2\gamma}G_0\bar{a}}{\Delta - i\gamma}\hat{x}(\omega). \tag{10}$$

Accompanying these input fluctuations is a classical mean amplitude, $\bar{a}_{in}$, and we can define a phase reference for the system by setting this field to be real and positive. This field also receives a phase shift upon interaction with the cavity:

$$\bar{a}_{out} = -\bar{a}_{in} + \sqrt{2\gamma}\bar{a} = -\frac{\Delta + i\gamma}{\Delta - i\gamma}\bar{a}_{in}. \tag{11}$$

The power fluctuation measured by a photodetector placed at the cavity output reads $\delta\hat{P} \equiv \bar{a}^*_{out}\hat{a}_{out} + \bar{a}_{out}\hat{a}^\dagger_{out}$. In our stated limit and in the frequency domain, this fluctuating piece is given by

$$\delta\hat{P}(\omega) = \bar{a}^*_{out}\hat{a}_{out}(\omega) + \bar{a}_{out}\hat{a}^\dagger_{out}(\omega) \approx \sqrt{2}\bar{a}_{in}\hat{v}_1(\omega). \tag{12}$$

Therefore, due to the common phase rotation experienced by the dc and fluctuating components [cf. Eqs. (10) and (11)], the output power is still a measure of the amplitude fluctuations of the input field. As shown in Eq. (7), it is this quadrature responsible for the radiation pressure back-action on the resonator, and so the noise can be suppressed by feeding this signal back to the amplitude of the pump laser.

IV. RESIDUAL RADIATION PRESSURE NOISE

While strong radiation pressure noise cancellation can be achieved using this technique, a small fraction cannot be canceled owing to two effects: (i) *optical loss* due to imperfection of the cavity and nonunity quantum efficiency in photodetection, which will introduce vacuum noise that is uncorrelated with $\hat{v}_1$ and $\hat{v}_2$; and (ii) *finite cavity bandwidth and detuning*, which modifies the input-output relation to give residual parasitic sensitivity to the oscillator displacement $\hat{x}$, which we have thus far ignored by assuming very large bandwidth and detuning. In actual experimental setups, there is always a certain amount of optical loss, and the bandwidth and detuning are both finite.

By taking these effects into account, we see that the total measured power fluctuation is

$$\delta\hat{P}(\omega) = \sqrt{2}\bar{a}_{in}\hat{v}_1(\omega) + \delta\hat{P}_\varepsilon(\omega) + \delta\hat{P}_\eta(\omega) + \delta\hat{P}_x(\omega). \tag{13}$$

Here, the second term,

$$\delta\hat{P}_\varepsilon(\omega) = \frac{2\sqrt{2\gamma\gamma_\varepsilon}\bar{a}_{in}}{\gamma^2 + \Delta^2}(\gamma\hat{v}'_1 - \Delta\hat{v}'_2), \tag{14}$$

arises from the vacuum fluctuation $\hat{v}'_{1,2}$ (uncorrelated with $\hat{v}_{1,2}$) due to optical loss in the cavity, and $\gamma_\varepsilon \equiv c\,\varepsilon/(4L)$, with ε being the round-trip optical loss in the cavity; the third term,

$$\delta\hat{P}_\eta(\omega) \approx \sqrt{2}\bar{a}_{in}\sqrt{1-\eta}\,\hat{n}, \tag{15}$$

comes from the nonunity quantum efficiency, η, of the photodetector (here, $\hat{n}$ is the vacuum fluctuation associated with this loss port); the last term represents the parasitic position sensitivity,

$$\delta\hat{P}_x(\omega) = -\frac{2G_0|\bar{a}|^2\Delta(2\gamma_\varepsilon - i\omega)}{\gamma^2 + \Delta^2}\hat{x}(\omega), \tag{16}$$

which arises both from the intracavity loss and from the first-order correction to the frequency dependence due to finite bandwidth and detuning, and the associated quantum back-action (radiation pressure) noise reads

$$\hat{F}_{\delta\hat{P}_x} = -\frac{2\hbar G_0\sqrt{\gamma}|\bar{a}|}{\sqrt{\gamma^2 + \Delta^2}}\left[\sqrt{\frac{\gamma_\varepsilon}{\gamma}}\,\hat{v}'_1 + \frac{i\omega\Delta}{\gamma^2 + \Delta^2}\hat{v}_2\right]. \tag{17}$$

Using this modified photodetector output, we compute a full, closed-loop noise model of the system (details in the Appendix). In the ideal feedback limit—i.e., for infinite open-loop gain—the residual force noise

$$\hat{F}^{res}|_{gain\to\infty} = \frac{2\hbar G_0\sqrt{\gamma}|\bar{a}|}{\sqrt{\gamma^2 + \Delta^2}}\left\{\sqrt{\frac{\gamma_\varepsilon}{\gamma}}\frac{(\gamma^2 - \Delta^2)\hat{v}'_1 + 2\gamma\Delta\hat{v}'_2}{\gamma^2 + \Delta^2} - \sqrt{1-\eta}\,\hat{n} - \frac{i\,\omega\Delta\hat{v}_2}{\gamma^2 + \Delta^2}\right\}, \tag{18}$$

of which the spectral density reads

$$S_F^{res} = \frac{4\hbar^2 G_0^2\gamma|\bar{a}|^2}{\gamma^2 + \Delta^2}\left[\frac{\gamma_\varepsilon}{\gamma} + (1-\eta) + \frac{\omega^2\Delta^2}{(\gamma^2 + \Delta^2)^2}\right]. \tag{19}$$

By comparison with the thermal force spectrum from a viscous damping model, $S_F^{th} = 4m\gamma_m k_B T$, we can assign an effective

TABLE I. A sample set of parameters. These values generate an optical spring with $\omega_{os}/2\pi \approx 100$ kHz and $Q_{eff} \approx 10^9$. The laser powers P_B and P_R refer to the circulating powers, and Q refers to the quality of the bare mechanical system. For the specified geometry, the required finesses are of order $\mathscr{F} \approx 10\,000$, compatible with the optical quality of resonators in production today.

	Parameter									
	m	L	$\omega_m/2\pi$	Q	$\gamma_B/2\pi$	$\Delta_B/2\pi$	P_B	$\gamma_R/2\pi$	$\Delta_R/2\pi$	P_R
Value	250 ng	1 mm	100 Hz	10^6	20 MHz	−20 MHz	390 mW	4 MHz	4 MHz	16 mW

temperature to this residual force noise as

$$T_{\text{eff}}^{\text{res}} \equiv \frac{S_F^{\text{res}}}{4m\gamma_m k_B}. \tag{20}$$

In order not to dominate, this residual temperature must be kept below the environmental temperature.

Another interesting result of this closed-loop analysis is that, again for an infinite loop gain, the effective mechanical susceptibility of the resonator becomes

$$\chi_{\text{eff}}^{-1} \to \chi_{\text{eff}'}^{-1} = -m\left[\omega^2 + i\gamma_m\omega - \left(\omega_m^2 + \omega_{os}^2\right)\right]. \tag{21}$$

Comparing this with Eq. (5), we see that the damping contribution from the optical spring, Γ_{os}, is removed. Recall that, for an optical spring with a positive restoring force, we have negative damping, $\Gamma_{os} < 0$ [cf. Eq. (4)], and it is for this reason that a second optical spring field is usually necessary to make the system stable—the double-optical-spring scheme [26], discussed in the next section. In our case, if the loop gain is high enough (i.e., if $G_{OL} \gg |\Gamma_{os}|/|\gamma_m|$), the negative damping will be removed due to the finite response to the mechanical displacement, indicated by Eq. (16), and therefore the system can be stabilized by the positive internal damping of the mechanical system. A practical issue in implementing this is that the required gain could be high in certain applications, and a double optical spring can therefore be used to ease the requirement.

V. EXPERIMENTAL REALIZATION WITH A DOUBLE OPTICAL SPRING

In the following, we detail a proposed experiment in which a mechanical oscillator with a bare resonance frequency of $\omega_m/2\pi = 100$ Hz is optically stiffened to a new, optomechanical resonance of $\omega_{os}/2\pi \approx 100$ kHz, leading to a commensurate increase in its effective Q factor. This parameter regime is chosen because—due to the low natural loss rate of the resonator—it highlights the long thermal decoherence time scales achievable with such a technique.

Despite the active stabilization effect discussed above, it may be impractical to use a single optical spring due to the very high feedback gains required.[1] Instead, we consider a novel approach proposed in Ref. [26] that uses a second optical spring field to create a passively stable system. The linear combination of two K_{os}'s, with one red detuned and the other blue, can be made to exhibit both positive restoring and damping, resulting in a passively stable spring. The sum of the contributions from both optical springs is

$$K_{os}^{\text{tot}} \approx -im\omega\left[\frac{\gamma_B\omega_{os_B}^2}{\left(\gamma_B^2 + \Delta_B^2\right)} - \frac{\gamma_R\omega_{os_R}^2}{\left(\gamma_R^2 + \Delta_R^2\right)}\right] + m\omega_{os_B}^2 - m\omega_{os_R}^2 \tag{22}$$

where γ_B, γ_R and Δ_B, Δ_R are the cavity bandwidth and detuning as seen by the blue and red fields, respectively (note that $\Delta_B < 0$). For proper choice of these parameters as a function of the ratio $|\omega_{os_B}/\omega_{os_A}| > 1$, the expression in brackets can be made to vanish, and the effective resonator is stiffened without instability or excess damping.[2] Additionally, the effect of the feedback discussed above is to suppress the damping contribution from both springs, causing any mismatching of the damping cancellation to be further suppressed. In practice, it may not be trivial to set different bandwidths for two optical fields of macroscopically similar frequency. In this case (i.e., $\gamma_B = \gamma_R \equiv \gamma$), one can still cancel the imaginary terms by choosing the appropriate detunings. In particular, if $|\omega_{os_B}/\omega_{os_A}| = \kappa$, cancellation is obtained for $\Delta_B^2 = (\kappa^2 - 1)\gamma^2 + \kappa^2\Delta_R^2$.

A set of sample parameters is listed in Table I. Under these conditions, an oscillator with a resonant frequency of $\omega_{os}/2\pi \approx 100$ kHz and an effective Q of 10^9 is formed.[3] Such a device can, in principle, be cooled to its ground state from an environmental temperature of $T \approx 4800$ K (clearly, this should not be attempted, but it serves to illustrate what this technique implies in the context of quantum experiments)! From Eq. (20), we can also calculate the effective temperatures of the residual quantum-radiation-pressure noise from the two

[1] In our example below, using a single optical spring would dictate optical damping Γ_{os} of the order of $2\pi \times 1$ kHz. The mechanical damping is $\gamma_m \approx 2\pi \times 10^{-4}$ Hz, and therefore the required gain at the optical spring frequency of 100 kHz is $G_{OL}^{\text{req}} \approx 10^5$. In practice, obtaining laser amplitude actuation bandwidths above ∼1 MHz is quite challenging, and so it would be difficult to implement a stable loop in this case.

[2] Note that the expression need not vanish, but only be positive for the resultant resonator to be stable. Furthermore, any positive damping from the optical fields is cold and, therefore, does not contribute noise or degrade the signal-to-noise ratio. We specifically consider the case of zero additional damping, however, since it leads to an effective resonator whose Q is determined solely by the intrinsic damping of the bare mechanical system.

[3] This Q_{eff} value is calculated assuming a viscous damping model; the mechanical damping, γ_m, is fixed, and so, since the optical spring adds no damping, the improvement is given by $Q_{\text{eff}} = (\omega_{os}/\omega_m)Q$. Several candidate mechanical resonators are predicted to be better approximated by a structural damping model, in which case the improvement in Q is potentially much greater.

optical spring fields as $T_{\rm eff}^{\rm res,B} = T_{\rm eff}^{\rm res,R} \approx 23$ mK, in the lossless case, and $T_{\rm eff}^{\rm res,B} \approx 84$ K and $T_{\rm eff}^{\rm res,R} \approx 60$ K for realistic losses: 99% quantum efficiency [34] and $\varepsilon = 30$ ppm [35]. Even in the lossy case, the residual noise temperatures are considerably lower than most target environment temperatures.

VI. CONCLUSION

We have proposed a method for creating a tunable effective mechanical resonator with an extremely high Qf product. In addition, these resonators can be made to operate in lower frequency bands than current ones of competitive quality, allowing for exceptionally long rethermalization time scales. While the use of optical dilution to mitigate thermal noise has been proposed and demonstrated in the past, we have considered a parameter regime in which the deleterious effects of quantum-radiation-pressure noise from strong optical spring fields can be all but eliminated, allowing for greatly unhindered dilution. We feel that the application of this technique holds great promise for any field requiring very-high-Q resonators, including, but not limited to, those of quantum optomechanics and sensitive force measurement.

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APPENDIX: DETAILED ANALYSIS OF THE SYSTEM

In this Appendix, we show some additional details for the derivation of the formulas presented in thetext. We first consider the ideal case without optical loss and show the leading-order terms in the large bandwidth and detuning limit. Then, we show the effect of optical loss and next-order correction terms. Finally, we consider the implementation of feedback and the closed-loop response of the system, which is relevant to actual experimental realization. Our notation here is nearly identical to that in Ref. [36].

1. Ideal situation: No optical loss and leading-order terms

In this section, we consider the ideal situation for a typical optomechanical device, which has been extensively covered in the literature [33,36–39]. We start with the standard Hamiltonian for the canonical optomechanical device, shown in the dashed box in Fig. 1:

$$\hat{\mathscr{H}} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_m^2\hat{x}^2 + \hbar\omega_c\hat{a}^\dagger\hat{a} + \hbar G_0\hat{x}\hat{a}^\dagger\hat{a} + i\hbar\sqrt{2\gamma}[\hat{a}_{\rm ext}(t)e^{-i\omega_0 t}\hat{a}^\dagger - \hat{a}_{\rm ext}^\dagger(t)e^{i\omega_0 t}\hat{a}]. \quad \text{(A1)}$$

Here, the first two terms are the free Hamiltonian for the oscillator, with ω_m being the mechanical frequency; the third term is the free Hamiltonian for the cavity mode (ω_c is the cavity resonant frequency, and $\hat{a}$ is its annihilation operator satisfying $[\hat{a}, \hat{a}^\dagger] = 1$); the fourth term describes the interaction between the oscillator and the cavity mode, with $G_0 = \omega_c/L$ being the coupling strength and L the cavity length; the remaining part is the coupling between the cavity mode and the external continuum $\hat{a}_{\rm ext}(t)$, with coupling rate γ and $[\hat{a}_{\rm ext}(t), \hat{a}_{\rm ext}^\dagger(t')] = \delta(t-t')$, from which one can define the input operator $\hat{a}_{\rm in}$ (ingoing before interaction) and output operator $\hat{a}_{\rm out}$ (outgoing after interaction) through

$$\hat{a}_{\rm in} \equiv \hat{a}_{\rm ext}(t_-), \quad \hat{a}_{\rm out} \equiv \hat{a}_{\rm ext}(t_+), \quad \text{(A2)}$$

according to the standard input-output formalism [40]. In the Hamiltonian, we have ignored terms accounting for the dissipation mechanism of the mechanical oscillator coupling to its thermal environment. We later include their effects in the equation of motion for the oscillator.

a. Linearized Hamiltonian

In the experiment, the cavity mode is driven coherently by a laser with a high amplitude at frequency ω_0. We can therefore study the linearized dynamics by perturbing around the steady state. In the rotating frame of the laser frequency ω_0, the corresponding linearized Hamiltonian for the system reads

$$\hat{\mathscr{H}} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_m^2\hat{x}^2 + \hbar\Delta\hat{a}^\dagger\hat{a} + \hbar G_0\hat{x}(\bar{a}^*\hat{a} + \bar{a}\,\hat{a}^\dagger) + i\hbar\sqrt{2\gamma}[\hat{a}_{\rm ext}(t)\hat{a}^\dagger - \hat{a}_{\rm ext}^\dagger(t)\hat{a}]. \quad \text{(A3)}$$

Here, the cavity detuning is the difference between the cavity resonant frequency and the laser frequency (i.e., $\Delta \equiv \omega_c - \omega_0$). $\bar{a}$ is the steady-state amplitude of the cavity mode, and if we choose the phase reference such that the steady-state amplitude of the input field, $\bar{a}_{\rm in}$, is real and positive, we have

$$\bar{a} = \frac{\sqrt{2\gamma}\,\bar{a}_{\rm in}}{\gamma + i\Delta} = \frac{\sqrt{2\gamma}}{\gamma + i\Delta}\sqrt{\frac{P_{\rm in}}{\hbar\omega_0}}, \quad \text{(A4)}$$

where $P_{\rm in}$ is the input laser power. These operators in the above Hamiltonian should be viewed as perturbed parts of the original ones, and the quantum state they act on is also transformed correspondingly. For instance, the input state for $\hat{a}_{\rm in}$ is originally a coherent state (for an ideal laser), and now it is the vacuum state $|0\rangle$ with

$$\langle 0|\hat{a}_{\rm in}(t)\hat{a}_{\rm in}^\dagger(t')|0\rangle = \delta(t-t'). \quad \text{(A5)}$$

b. Equations of motion

Given the above Hamiltonian, the cavity mode satisfies the Heisenberg equation of motion

$$\dot{\hat{a}}(t) + (\gamma + i\Delta)\hat{a}(t) = -iG_0\bar{a}\,\hat{x}(t) + \sqrt{2\gamma}\,\hat{a}_{\rm in}(t), \quad \text{(A6)}$$

and it is related to the cavity output $\hat{a}_{\rm out}$ by the standard input-output relation:

$$\hat{a}_{\rm out}(t) = -\hat{a}_{\rm in}(t) + \sqrt{2\gamma}\,\hat{a}(t). \quad \text{(A7)}$$

Similarly, we can read off the equation of motion for the oscillator:

$$m[\ddot{\hat{x}}(t) + \gamma_m\dot{\hat{x}}(t) + \omega_m^2\hat{x}(t)] = \hat{F}_{\rm rad}(t) + \hat{F}_{\rm th}(t). \quad \text{(A8)}$$

Here, we have defined the radiation pressure,

$$\hat{F}_{\rm rad}(t) \equiv -\hbar G_0[\bar{a}^*\hat{a}(t) + \bar{a}\,\hat{a}^\dagger(t)]. \tag{A9}$$

In addition, we have added the damping term $m\gamma_m\dot{\hat{x}}(t)$ and the associated thermal fluctuation force $\hat{F}_{\rm th}$ into the equation of motion, of which the correlation function is $\langle\hat{F}_{\rm th}(t)\hat{F}_{\rm th}(t')\rangle = 4m\gamma_m k_B T\delta(t-t')$ in the high-temperature limit $k_B T \gg \hbar\omega_m$.

We note here that the equations of motion we have derived are formally identical to those in the classical case. The quantum noise can also be described quasiclassically using a Poisson statistical approach, however, we choose to use the quantum formalism for two reasons: (i) It dramatically simplifies the analysis in cases such as these with multiple loss channels; and (ii) despite the absence of explicitly nonclassical photonic states, the effects we describe here are fundamentally quantum mechanical in nature.

c. Solution for the cavity mode

The above linear equations of motion can be solved in the frequency domain. The solution for the cavity mode reads

$$\hat{a}(\omega) = \frac{G_0\bar{a}\,\hat{x}(\omega) + i\sqrt{2\gamma}\,\hat{a}_{\rm in}(\omega)}{\omega - \Delta + i\gamma}. \tag{A10}$$

From this, we can obtain the expression for the radiation pressure:

$$\hat{F}_{\rm rad}(\omega) = -K_{\rm os}(\omega)\hat{x}(\omega) + \hat{F}_{\rm noise}(\omega). \tag{A11}$$

We introduce the optical spring coefficient $K_{\rm os}$ as

$$K_{\rm os}(\omega) \equiv \frac{2\hbar G_0^2|\bar{a}|^2\Delta}{(\omega-\Delta+i\gamma)(\omega+\Delta+i\gamma)} \tag{A12}$$

and the quantum-radiation-pressure noise term as

$$\hat{F}_{\rm noise}(\omega) \equiv \frac{2\hbar G_0|\bar{a}|\sqrt{\gamma}}{\sqrt{\gamma^2+\Delta^2}}\left[\frac{(\gamma^2+\Delta^2-i\,\gamma\,\omega)\hat{v}_1 + i\,\Delta\,\omega\,\hat{v}_2}{(\omega-\Delta+i\gamma)(\omega+\Delta+i\gamma)}\right], \tag{A13}$$

with $\hat{v}_1 \equiv (\hat{a}_{\rm in} + \hat{a}_{\rm in}^\dagger)/\sqrt{2}$ and $\hat{v}_2 \equiv (\hat{a}_{\rm in} - \hat{a}_{\rm in}^\dagger)/\sqrt{2}i$ being the vacuum fluctuation of the input amplitude and phase quadratures, respectively. The strength of the radiation pressure noise can be quantified by its power spectrum, which is defined through

$$\langle 0|\hat{F}_{\rm noise}^\dagger(\omega)\hat{F}_{\rm noise}(\omega')|0\rangle_{\rm sym} \equiv \pi\, S_F(\omega)\delta(\omega-\omega'), \tag{A14}$$

where the subscript "sym" denotes for symmetrization and the spectrum is a single-sided one. Note that for the vacuum input state $\langle 0|\hat{v}_k^\dagger(\omega)\hat{v}_l(\omega')|0\rangle_{\rm sym} = \pi\,\delta_{kl}\,\delta(\omega-\omega')$, and therefore

$$S_F(\omega) = \frac{4\hbar^2 G_0^2|\bar{a}|^2\gamma(\gamma^2+\omega^2+\Delta^2)}{[(\omega-\Delta)^2+\gamma^2][(\omega+\Delta)^2+\gamma^2]}. \tag{A15}$$

For the case of large bandwidth and detuning in which we are interested, the above radiation pressure noise can be approximated as (up to zeroth order of ω)

$$\hat{F}_{\rm noise}(\omega) \approx -\frac{2\hbar G_0|\bar{a}|\sqrt{\gamma}}{\sqrt{\gamma^2+\Delta^2}}\hat{v}_1(\omega) \propto \hat{v}_1(\omega). \tag{A16}$$

This indicates that the quantum-radiation-pressure noise is mostly contributed by fluctuations in the amplitude quadrature of the input field. It can be directly measured at the cavity output using a photodetector, as we see later; this is why we can evade such noise by feeding back with an appropriate linear filter, which is the central idea of this work.

d. Solution for the mechanical oscillator

Given the expression for the radiation pressure, we can write down the solution for the mechanical displacement $\hat{x}$ as

$$\hat{x}(\omega) = \frac{\hat{F}_{\rm noise}(\omega) + \hat{F}_{\rm th}(\omega)}{-m\left[\omega^2 - \omega_m^2 + i\gamma_m\omega\right] + K_{\rm os}(\omega)}. \tag{A17}$$

As we can see, the mechanical susceptibility is modified into an effective one due to the optical spring effect. Since we are focusing on the case of large cavity bandwidth and detuning, the optical spring response $K_{\rm os}$ can be expanded as

$$\begin{aligned} K_{\rm os} &\approx -\frac{2\hbar G_0^2|\bar{a}|^2\Delta}{\Delta^2+\gamma^2}\left[1 + \frac{2i\gamma\omega}{\Delta^2+\gamma^2}\right] \\ &\equiv m\omega_{\rm os}^2 - i\,m\Gamma_{\rm os}\omega, \end{aligned} \tag{A18}$$

where $\omega_{\rm os}$ is the optical spring frequency and $\Gamma_{\rm os}$ is the optical damping coefficient. We can then rewrite the mechanical displacement $\hat{x}$ as

$$\hat{x}(\omega) = \chi_{\rm eff}(\omega)[\hat{F}_{\rm noise}(\omega) + \hat{F}_{\rm th}(\omega)], \tag{A19}$$

where the effective mechanical susceptibility $\chi_{\rm eff}$ is defined through

$$\chi_{\rm eff}^{-1}(\omega) \equiv -m\left[\omega^2 + i(\gamma_m + \Gamma_{\rm os})\omega - \left(\omega_m^2 + \omega_{\rm os}^2\right)\right]. \tag{A20}$$

In the negative-detuning case $\Delta < 0$, $\omega_{\rm os}$ is positive and real, and the damping $\Gamma_{\rm os}$ is negative; in the positive-detuning case $\Delta > 0$, $\omega_{\rm os}$ is purely imaginary and the damping $\Gamma_{\rm os}$ is positive. In both cases, the mechanical system is potentially unstable, especially when the intrinsic damping γ_m is small as in our proposed parameter regime. By introducing an additional laser with a different detuning frequency, we can combine two optical springs and achieve both positive rigidity and damping—the so-called double optical spring. Such a scheme has been realized experimentally by Corbitt *et al.* [26]. We can therefore significantly upshift the mechanical resonant frequency while keeping the oscillator stable.

e. Solution for the cavity output

From the input-output relation, the cavity output is given by

$$\hat{a}_{\rm out}(\omega) = -\frac{\omega-\Delta-i\gamma}{\omega-\Delta+i\gamma}\hat{a}_{\rm in}(\omega) + \frac{\sqrt{2\gamma}\,G_0\bar{a}}{\omega-\Delta+i\gamma}\hat{x}(\omega). \tag{A21}$$

In the limit of high bandwidth and detuning, we can approximate this as

$$\hat{a}_{\rm out}(\omega) = -\frac{\Delta+i\gamma}{\Delta-i\gamma}\hat{a}_{\rm in}(\omega) - \frac{\sqrt{2\gamma}G_0\bar{a}}{\Delta-i\gamma}\hat{x}(\omega). \tag{A22}$$

Similarly, for the classical amplitude at dc, we have the input-output relation

$$\bar{a}_{\rm out} = -\frac{\Delta+i\gamma}{\Delta-i\gamma}\bar{a}_{\rm in}. \tag{A23}$$

The photodetector measures the power of the cavity output field:

$$\hat{P}_{\rm out}(t) = |(\bar{a}^*_{\rm out} + \hat{a}^\dagger_{\rm out})(\bar{a}_{\rm out} + \hat{a}_{\rm out})| = |\bar{a}_{\rm out}|^2 + \delta\hat{P}(t) + \hat{a}^\dagger_{\rm out}\hat{a}_{\rm out}. \quad \text{(A24)}$$

It contains the classical dc part $|\bar{a}_{\rm out}|^2$ and the leading-order time-varying component

$$\delta\hat{P}(t) \equiv \bar{a}^*_{\rm out}\hat{a}_{\rm out} + \bar{a}_{\rm out}\hat{a}^\dagger_{\rm out} \quad \text{(A25)}$$

that we are interested in, which, in the frequency domain, is given by

$$\delta\hat{P}(\omega) = \bar{a}^*_{\rm out}\hat{a}_{\rm out}(\omega) + \bar{a}_{\rm out}\hat{a}^\dagger_{\rm out}(\omega) \approx \sqrt{2}\,\bar{a}_{\rm in}\hat{v}_1(\omega). \quad \text{(A26)}$$

This means that the photodetector mostly measures fluctuations in the amplitude quadrature of the input field, which is the main contributor to the quantum-radiation-pressure noise felt by the mechanical oscillator as shown by Eq. (A16). Therefore, simply by feeding back the photodetector signal to the mechanical oscillator, we will be able to evade the quantum-radiation-pressure noise. The only limitation arises from the optical loss and the frequency dependence of the radiation pressure noise that we have ignored in Eq. (A16) by assuming a large cavity bandwidth and detuning.

2. Realistic situation: Optical loss and next-order corrections

In this section, we analyze the effect of optical loss and also the next-order correction—frequency dependence of the radiation pressure noise as well as nonzero response to the mechanical displacement in the photocurrent—due to the finite cavity bandwidth. As mentioned in the text, the optical loss will decrease the noise cancellation efficiency by introducing vacuum fluctuations—which we denote $\hat{a}'_{\rm in}$—that are uncorrelated with $\hat{a}_{\rm in}$. In terms of the equation of motion for the cavity mode, we have

$$\dot{\hat{a}} + (\gamma_{\rm tot} + i\Delta)\hat{a} = -iG_0\bar{a}\hat{x} + \sqrt{2\gamma}\,\hat{a}_{\rm in} + \sqrt{2\gamma_\varepsilon}\,\hat{a}'_{\rm in}, \quad \text{(A27)}$$

where $\bar{a}$ is modified into

$$\bar{a} = \frac{\sqrt{2\gamma}\,\bar{a}_{\rm in}}{\gamma_{\rm tot} + i\Delta} \quad \text{(A28)}$$

and we have introduced

$$\gamma_{\rm tot} \equiv \gamma + \gamma_\varepsilon = \gamma + c\,\varepsilon/(4L), \quad \text{(A29)}$$

where ε is the round-trip power loss factor in the cavity.

a. Modification of the radiation pressure

Correspondingly, this will modify the radiation pressure [cf. Eq. (A11)],

$$\hat{F}_{\rm rad}(\omega) = -K_{\rm os}(\omega)\hat{x}(\omega) + \hat{F}_{\rm noise}(\omega), \quad \text{(A30)}$$

where

$$K_{\rm os} = \frac{2\hbar G_0^2|\bar{a}|^2\Delta}{(\omega - \Delta + i\gamma_{\rm tot})(\omega + \Delta + i\gamma_{\rm tot})} \quad \text{(A31)}$$

and

$$\hat{F}_{\rm noise} \equiv \frac{2\hbar G_0\sqrt{\gamma}|\bar{a}|}{\sqrt{\gamma_{\rm tot}^2 + \Delta^2}}\left\{\frac{(\gamma_{\rm tot}^2 + \Delta^2 - i\,\gamma_{\rm tot}\,\omega)\hat{v}_1 + i\,\Delta\,\omega\,\hat{v}_2}{(\omega - \Delta + i\gamma_{\rm tot})(\omega + \Delta + i\gamma_{\rm tot})} + \sqrt{\frac{\gamma_\varepsilon}{\gamma}}\frac{(\gamma_{\rm tot}^2 + \Delta^2 - i\,\gamma_{\rm tot}\,\omega)\hat{v}'_1 + i\,\Delta\,\omega\,\hat{v}'_2}{(\omega - \Delta + i\gamma_{\rm tot})(\omega + \Delta + i\gamma_{\rm tot})}\right\}, \quad \text{(A32)}$$

where $\hat{v}'_1 \equiv (\hat{a}'_{\rm in} + \hat{a}'^\dagger_{\rm in})/\sqrt{2}$ and $\hat{v}'_2 \equiv (\hat{a}'_{\rm in} - \hat{a}'^\dagger_{\rm in})/(\sqrt{2}i)$.

Again, for large bandwidth and detuning, and keeping up to the next-order correction—leading order of ε and ω—we obtain

$$K_{\rm os} = -\frac{2\hbar G_0^2|\bar{a}|^2\Delta}{\Delta^2 + \gamma^2}\left[1 - \frac{4\gamma\gamma_\varepsilon}{\gamma^2 + \Delta^2} + \frac{2i\gamma\omega}{\gamma^2 + \Delta^2}\right] \quad \text{(A33)}$$

and

$$\hat{F}_{\rm noise} = -\frac{2\hbar G_0\sqrt{\gamma}|\bar{a}|}{\sqrt{\gamma^2 + \Delta^2}}\left[\hat{v}_1 + \frac{i\omega\Delta}{\gamma^2 + \Delta^2}\hat{v}_2 + \sqrt{\frac{\gamma_\varepsilon}{\gamma}}\,\hat{v}'_1\right]. \quad \text{(A34)}$$

b. Modification of the input-output relation

Similarly, the input-output relation is also modified into

$$\hat{a}_{\rm out} = -\frac{\omega - \Delta - i(\gamma - \gamma_\varepsilon)}{\omega - \Delta + i\gamma_{\rm tot}}\hat{a}_{\rm in} + \frac{2i\sqrt{\gamma\gamma_\varepsilon}}{\omega - \Delta + i\gamma_{\rm tot}}\hat{a}'_{\rm in} + \frac{\sqrt{2\gamma}\,G_0\bar{a}}{\omega - \Delta + i\gamma_{\rm tot}}\hat{x}. \quad \text{(A35)}$$

c. Modification of the photocurrent output

The exact expression for the ac part of the photocurrent output $\delta\hat{P}(\omega)$ is quite complicated; however, in our stated limit, we have

$$\delta I(\omega) \equiv \sqrt{2}\bar{a}_{\rm in}\hat{v}_1(\omega) + \delta\hat{P}_\varepsilon(\omega) + \delta\hat{P}_\eta(\omega) + \delta\hat{P}_x(\omega), \quad \text{(A36)}$$

where the term $\delta\hat{P}_\varepsilon$ contains the vacuum fluctuations $\hat{v}_{1,2}$ that are associated with optical loss:

$$\delta\hat{P}_\varepsilon(\omega) = \frac{2\sqrt{2\gamma\gamma_\varepsilon}\bar{a}_{\rm in}}{\gamma^2 + \Delta^2}(\gamma\,\hat{v}'_1 - \Delta\,\hat{v}'_2). \quad \text{(A37)}$$

The additional noise term $\delta\hat{P}_\eta$, due to the nonunity quantum inefficiency η of the photodetector (keeping to the first order of small $1 - \eta$), is

$$\delta\hat{P}_\eta(\omega) \approx \sqrt{2}\bar{a}_{\rm in}\sqrt{1 - \eta}\,\hat{n}, \quad \text{(A38)}$$

and the term $\delta\hat{P}_x$ depends on the mechanical displacement:

$$\delta\hat{P}_x(\omega) = -\frac{2G_0|\bar{a}|^2\Delta(2\gamma_\varepsilon - i\omega)}{\gamma^2 + \Delta^2}\hat{x}(\omega). \quad \text{(A39)}$$

Therefore, not only is there excess noise from the vacuum fluctuations introduced by the optical loss and nonunity quantum efficiency, but also there is a parasitic sensitivity to mechanical displacement, which is actually associated with the excess radiation pressure [cf. Eq. (A34)], compared with the ideal case [cf. Eq. (A16)].

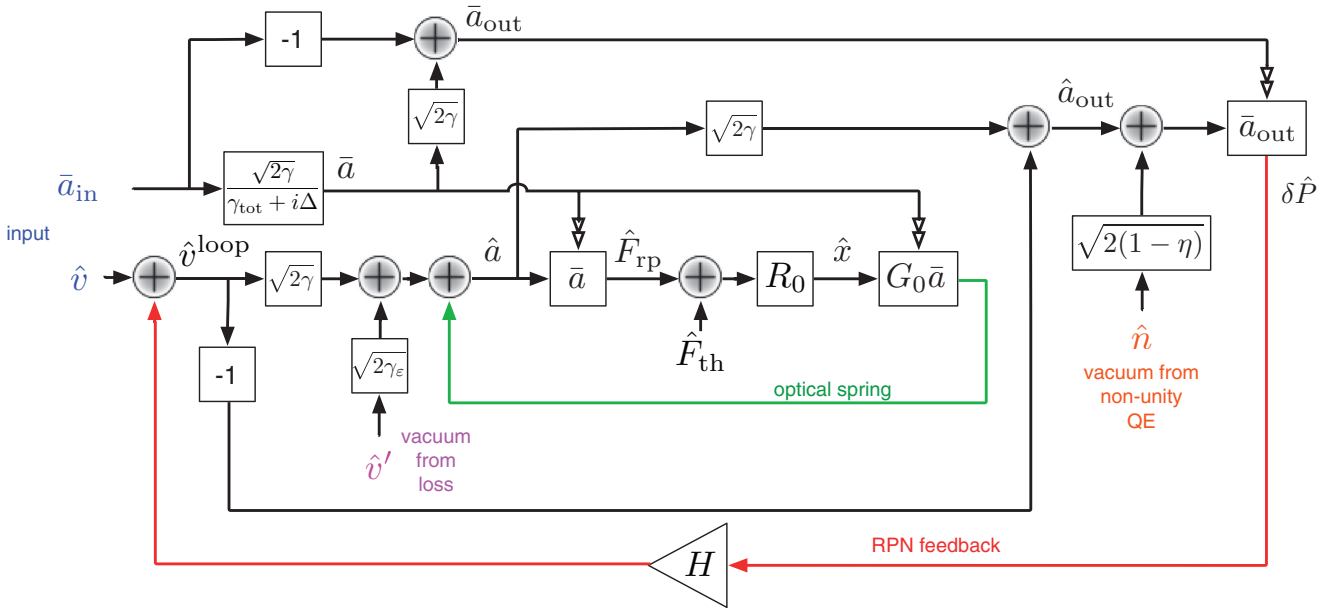

FIG. 2. (Color online) Graphical representation of the feedback model described in the text. The effects of the optical spring and the active external feedback are shown explicitly, while the (static) resonance of the classical field $\bar{a}$ is not. Using the input-output relations, the input fields at the left are split into the prompt reflections and the portions that enter the cavity. Then the leakage fields are summed with the prompt reflections to give the output. The feedback kernel is $H \equiv -K_c(\delta\hat{P}/\sqrt{2}\bar{a}_{\rm in})$. Note that double-headed arrows correspond to the setting of a parameter block, while single-headed arrows denote signal transmission as usual.

3. Feedback and closed-loop response

The radiation pressure noise can be removed either by feedforward (i.e., the photocurrent output is fed forward to the mechanical oscillator as a force) or by feedback (i.e., the photocurrent output is fed back to the input field via an amplitude modulator). Here, we consider the implementation of the feedback scheme. Not only is it more robust against uncertainty in the model transfer functions, but also, as we show, it can remove the negative damping in the optical spring and stabilize the mechanical oscillator, allowing, in principle, for a stable *single* optical spring.

According to the diagram shown in Fig. 2, the photocurrent output is fed back to an amplitude modulator, which modulates the amplitude quadrature of the input field. The set of equations for relevant quantities describe such a feedback scheme as follows, keeping up to the leading order of ε and ω:

$$\hat{x} = \chi_{\rm eff}(\hat{F}_{\rm noise} + \hat{F}_{\rm th}), \tag{A40}$$

$$\hat{F}_{\rm noise} = -\frac{2\hbar G_0\sqrt{\gamma}|\bar{a}|}{\sqrt{\gamma^2+\Delta^2}}\left[\hat{v}_1^{\rm loop} + \frac{i\omega\Delta}{\gamma^2+\Delta^2}\hat{v}_2 + \sqrt{\frac{\gamma_\varepsilon}{\gamma}}\,\hat{v}_1'\right], \tag{A41}$$

$$\delta\hat{P} = \sqrt{2}\bar{a}_{\rm in}\hat{v}_1^{\rm loop} + \delta\hat{P}_\varepsilon + \delta\hat{P}_\eta + \delta\hat{P}_x, \tag{A42}$$

$$\hat{v}_1^{\rm loop} = \hat{v}_1 - K_c(\delta\hat{P}/\sqrt{2}\bar{a}_{\rm in}). \tag{A43}$$

Here $\hat{v}_1^{\rm loop}$ is the in-loop amplitude quadrature after the amplitude modulator; K_c is the feedback kernel function, and we intentionally leave out the factor $\sqrt{2}\bar{a}_{\rm in}$ to simplify the equations.

We are interested in the motion of the mechanical oscillator when the feedback is turned on. Solving the above equations leads to

$$\hat{x} = \chi_{\rm eff'}(\hat{F}_{\rm noise'} + \hat{F}_{\rm th}), \tag{A44}$$

where

$$\chi_{\rm eff'}^{-1} = \chi_{\rm eff}^{-1} - \frac{4\hbar G_0^2|\bar{a}|^2\gamma\Delta(2\gamma_\varepsilon - i\omega)}{(\gamma^2+\Delta^2)^2}\frac{K_c}{1+K_c} \tag{A45}$$

and

$$\begin{aligned}\hat{F}_{\rm noise'} = -\frac{2\hbar G_0\sqrt{\gamma}|\bar{a}|}{\sqrt{\gamma^2+\Delta^2}}\Bigg[&\frac{1}{1+K_c}\hat{v}_1 \\ &-\sqrt{\frac{\gamma_\varepsilon}{\gamma}}\left(\frac{K_c-1}{K_c+1}\gamma^2-\Delta^2\right)\hat{v}_1' + \frac{2\sqrt{\gamma_\varepsilon\gamma}\,\Delta K_c}{K_c+1}\hat{v}_2' \\ &-\frac{K_c}{1+K_c}\sqrt{1-\eta}\,\hat{n} + \frac{i\omega\Delta}{\gamma^2+\Delta^2}\hat{v}_2\Bigg].\end{aligned} \tag{A46}$$

a. Ideal feedback limit

If we make $K_c \to \infty$, namely, in the ideal feedback limit, we have

$$\begin{aligned}\chi_{\rm eff'}^{-1}|_{K_c\to\infty} &= \chi_{\rm eff}^{-1} - \frac{4\hbar G_0^2|\bar{a}|^2\gamma\Delta(2\gamma_\varepsilon - i\omega)}{(\gamma^2+\Delta^2)^2} \\ &= \chi_0^{-1} + K_{\rm os} - \frac{4\hbar G_0^2|\bar{a}|^2\gamma\Delta(2\gamma_\varepsilon - i\omega)}{(\gamma^2+\Delta^2)^2} \\ &= -m\left[\omega^2 + i\gamma_m\omega - \left(\omega_m^2 + \omega_{\rm os}^2\right)\right],\end{aligned} \tag{A47}$$

where we have plugged in the expression for $\chi_{\rm eff}$ [cf. Eq. (A20)] and $K_{\rm os}$ [cf. Eq. (A33)]. Interestingly, the original negative damping $\Gamma_{\rm os}$ in $K_{\rm os}$ associated with the positive

rigidity is canceled out, and the mechanical oscillator is stabilized. Therefore, using this feedback scheme, the resultant oscillator is stable with a shifted resonant frequency:

$$\omega_m^{\rm new} = \sqrt{\omega_m^2 + \omega_{\rm os}^2}. \tag{A48}$$

Now, we quantify the residual radiation pressure noise on the mechanical oscillator. We have

$$\hat{F}'_{\rm noise}|_{K_c\to\infty} = \frac{2\hbar G_0\sqrt{\gamma}|\bar{a}|}{\sqrt{\gamma^2+\Delta^2}}\left\{\sqrt{\frac{\gamma_\varepsilon}{\gamma}}\frac{(\gamma^2-\Delta^2)\hat{v}'_1 + 2\gamma\Delta\hat{v}'_2}{\gamma^2+\Delta^2} - \sqrt{1-\eta}\,\hat{n} - \frac{i\,\omega\Delta\hat{v}_1}{\gamma^2+\Delta^2}\right\}. \tag{A49}$$

The corresponding spectral density reads

$$S_F^{\rm res} = \frac{4\hbar^2 G_0^2\gamma|\bar{a}|^2}{\gamma^2+\Delta^2}\left[\frac{\gamma_\varepsilon}{\gamma} + 1 - \eta + \frac{\omega^2\Delta^2}{(\gamma^2+\Delta^2)^2}\right]. \tag{A50}$$

The first term accounts for the effect of the optical loss, the second accounts for nonunity quantum efficiency of the photodetector, and the third term accounts for a finite cavity bandwidth.

4. Proposed experimental setup

While the technique described in this paper is quite general, a possible experimental layout is shown in Fig. 3. A laser's frequency is stabilized to the optical spring cavity length using the Pound-Drever-Hall (PDH) [41] locking technique. The laser is then detuned from the resonance by injecting an offset into the error point of the control loop. A second beam is picked off from the main laser and upshifted in frequency by an acousto-optic modulator (AOM). Once the detuning is set properly for both beams, the input power is ramped until the optical spring reaches the desired strength. At this point, the PDH frequency feedback to the laser can be disengaged, and—provided the mechanical and laser frequency stability are sufficiently high—the resonator is trapped in a passively stable potential by the optical spring forces.

Finally, the radiation pressure noise feedback described above is engaged, with the primary beam's signal fed back to the laser amplitude and the secondary's signal to the AOM drive amplitude. The electro-optic modulator (EOM), which imparts the phase modulation sidebands necessary for PDH lock, is also disengaged so as not to couple extra uncorrelated vacuum noise into the readout. In this operational

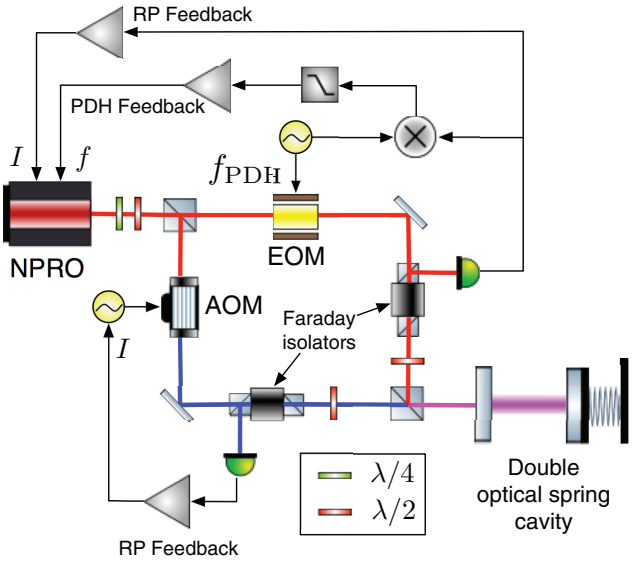

FIG. 3. (Color online) Proposed experimental setup. A laser beam is split and one path is upshifted in frequency, allowing for independent control of the detuning of each field. Each beam's intensity can also be controlled by feeding back to the laser or the modulator, and these channels are used for radiation pressure (RP) noise feedback. A Pound-Drever-Hall (PDH) locking scheme is used to set the operating point before strengthening the stable optical spring.

configuration, the quantum-radiation-pressure noise is very strongly suppressed, limited only by the parasitic loss and finite-bandwidth effects detailed above. Classical laser amplitude noise—which is indistinguishable from its quantum counterpart here—is also suppressed by the loop.

The only remaining potential issues are laser frequency stability and drift of the mechanical system, which can drive the optical spring fields away from their optimal detunings. The former can be avoided using prestabilization (e.g., by locking the laser to an external frequency reference). The latter is not as simple to avoid and will depend on the mechanical system in question. If necessary, a very weak PDH lock can be maintained using a low-frequency servo to ensure dc stability of the operating point. In this case, it may be possible to use weak enough control sideband fields that the RPN readout is still limited by the finite losses and bandwidth.

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